Journal Applied Mechanics and Technical Physics, Vol. 37, No. 1, 1996

CALCULATION OF PRESSURE DISTRIBUTION NEAR THE LEADING EDGE OF AN AIRFOIL USING THE DISCRETE-VORTEX METHOD

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UDC 518.12:533.6

Solution of problems of a flow past an airfoil using the discrete-vortex method determines the total intensities of a vortex layer for a finite number of elements. To calculate the pressure distribution over the airfoil and the total hydrodynamic reactions, one should construct an approximating function for the intensity of the vortex layer on the basis of assigned values of discrete vortices. A step function is usually chosen. Such an approximation gives good results for inner points of the contour, but is practically unacceptable near the leading edge, in particular, for thin profiles.

In the present work an approximating function for the intensity of a vortex layer near the leading edge of an airfoil is constructed taking into account the edge shape using the values for two discrete vortices. Formulas for the pressure distribution and total hydrodynamic reactions are presented. It is shown that in the limiting case of an infinitely thin profile the approximation permits exact determination of the suction force.

1. Let us consider the problem of a stationary flow of an ideal incompressible liquid past an airfoil in Cartesian coordinates Oxy. Let the contours L_1 and L_2 be the upper and lower sides of the airfoil. We model these contours by vortex layers of intensities $\gamma_1(\sigma_1)$ and $\gamma_2(\sigma_2)$ (σ_1 and σ_2 are the arc coordinates of the points at L_1 and L_2). Following the method of discrete vortices, we divide the contours L_1 and L_2 into N elements $[\zeta_{m-1}^{(r)}, \zeta_m^{(r)}] \in L_r$ ($r = 1, 2$, $m = 1, \ldots, N$), where $\zeta_{m-1}^{(r)}$ and $\zeta_m^{(r)}$ are the complex coordinates of the ends of the mth element. The total intensity of the vortex layer at $[\zeta_{m-1}^{(r)}, \zeta_m^{(r)}]$ determines the intensity of the discrete vortex $\Gamma_m^{(r)}$ placed at point $z_m^{(r)}$ of this element. Let the elements with number $m = 1$ be adjacent to the leading edge; the elements with number $m = N$, to the trailing edge.

Assume that $\Gamma_m^{(r)}$ are known. We construct the approximating functions for $\gamma_r(\sigma_r)$ at the elements $[\zeta_0^{(r)}, \zeta_1^{(r)}]$ through the assigned values of $\Gamma_1^{(r)}$ ($r = 1, 2$) with allowance for the profile geometry and the character of variation of fluid velocity along it.

First of all we derive a rather general equation for the profile contour in a small vicinity of the leading edge. We introduce a system of coordinates $O\xi\eta$ with the origin at the leading edge orienting the ξ axis along the tangent to the median L_0 of the contour (see Fig. 1). Let ξ_{r1}, η_{r1} be the coordinates of the end of the first element at L_r, $r = 1, 2$. We shall assign the contour L_r at the points of this element by the equation

$$\eta_r = (-1)^{r-1} a_r \sqrt{\xi}, \quad a_r = |\eta_{r1}|/\sqrt{\xi_{r1}}, \quad \xi \in [0, \xi_{r1}], \quad r = 1, 2. \tag{1.1}$$

Each curve (1.1) passes through two given points $(0,0)$, (ξ_{r1}, η_{r1}) of the contour L_r and satisfies the condition according to which the derivative

$$\eta_r' = (-1)^{r-1} a_r/(2\sqrt{\xi}) \tag{1.2}$$

Institute of Information Technologies and Applied Mathematics, Siberian Division, Russian Academy of Sciences, Omsk 644077. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 1, pp. 114–118, January–February, 1996. Original article submitted November 28, 1994; revision submitted January 4, 1995.

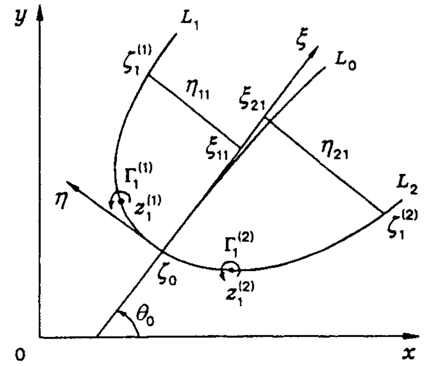

Fig. 1

becomes infinite at $\eta_{r1} \neq 0$ at the leading edge. Test calculations have demonstrated the high accuracy of Eq. (1.1) at $N \gg 1$ for all Zhukovskii airfoils (from a plate to a circle).

Now let us turn to the construction of an approximating function for the intensity of the vortex layer $\gamma_r(\sigma_r)$, which differs from the tangential velocity component only in the sign. We note that for round-leading-edge airfoils the fluid velocity along the edge changes rather smoothly, and for thin profiles a growth in the velocity gradient is observed as the leading edge is approached. In the limiting case of an infinitely thin profile, the velocity near the leading edge varies as $1/\sqrt{\xi}$. Taking into account this character of velocity variation we approximate $\gamma_r(\sigma_r)$ in the vicinity of the leading edge (at the elements $[\zeta_0^{(r)}, \zeta_1^{(r)}]$) by the function

$$\gamma_r(\sigma_r) = B_r + \frac{A_r}{\sqrt{\xi}\sqrt{1+(\eta_r')^2}}, \quad \xi \in [0, \xi_{r1}], \quad r = 1, 2,$$
$$A_r = \frac{\Delta_1^{(r)}}{4\sqrt{\xi_{r1}}}\left[\frac{\Gamma_1^{(1)}}{\Delta_1^{(1)}} + \frac{\Gamma_1^{(2)}}{\Delta_1^{(2)}}\right], \qquad B_r = \frac{1}{2}(-1)^{r-1}\left[\frac{\Gamma_1^{(1)}}{\Delta_1^{(1)}} - \frac{\Gamma_1^{(2)}}{\Delta_1^{(2)}}\right]. \tag{1.3}$$

Here $\Delta_1^{(r)}$ is the length of the element $[\zeta_0^{(r)}, \zeta_1^{(r)}]$; A_r and B_r are constants defined in terms of discrete vortices $\Gamma_1^{(1)}$ and $\Gamma_1^{(2)}$, which are placed at the first elements of the contours L_1 and L_2 (see Fig. 1).

Expressions (1.1) and (1.2) make it possible to write (1.3) as

$$\gamma_r(\sigma_r) = B_r + \frac{2A_r\sqrt{\xi_{r1}}}{\sqrt{4\xi\xi_{r1} + \eta_{r1}^2}}. \tag{1.4}$$

It is evident from (1.4) that for sufficiently thick profiles ($|\eta_{r1}|/\xi_{r1} \sim 1$) both terms exert the same influence on the velocity distribution, whereas for thin profiles ($|\eta_{r1}|/\xi_{r1} \ll 1$) the second term is principal. In the limiting case of $|\eta_{r1}| = 0$, when the airfoil coincides with the airfoil section L_0,

$$\gamma_r(\sigma_r) = A_r/\sqrt{\xi} + \dots, \qquad \xi \in [0, \xi_{r1}], \quad r = 1, 2.$$

In this case, L_0 is simulated by two vortex layers whose intensities γ_1 and γ_2 are determined (with one-place accuracy) by the limiting values of the tangential velocity components. In simulating L_0 by one vortex layer, as is customary in thin-wing theory [1], the intensity of the vortex layer is $\gamma = \gamma_1 + \gamma_2$ and in a small vicinity of the leading edge

$$\gamma(\xi) = A/\sqrt{\xi}, \quad A = A_1 + A_2. \tag{1.5}$$

This completes the construction of the approximating function for $\gamma_r(\sigma_r)$ near the leading edge of an airfoil.

A numerical experiment has shown that for the elements of vortex layers at L_1 and L_2 beyond the leading edge one can apply a piecewise constant intensity distribution $\gamma_r(\sigma_r)$. In this case

$$\gamma_r(\sigma_r) = \Gamma_m^{(r)}/\Delta_m^{(r)}, \quad \Delta_m^{(r)} = |\zeta_{m-1}^{(r)} - \zeta_m^{(r)}|, \quad r = 1, 2, \quad m = 2, \ldots, N. \tag{1.6}$$

2. Let us turn to the derivation of formulas for the hydrodynamic pressure p and the total hydrodynamic forces R_x and R_y acting on the profile contour $L = L_1 + L_2$.

In a stationary incompressible flow hydrodynamic pressure is related to the velocity vector $\mathbf{v}$ by Bernoulli's integral

$$p - p_\infty = -\frac{1}{2}\rho(v^2 - v_\infty^2)$$

(ρ is the liquid density, $v = |\mathbf{v}|$).

At the points of the profile contour

$$v = |\gamma_r|, \quad p - p_\infty = \frac{1}{2}\rho(\gamma_r^2 - v_\infty^2), \quad (x, y) \in L_r, \quad r = 1, 2. \tag{2.1}$$

Here $\gamma_r(\sigma_r)$ is determined from the assigned values for discrete vortices using approximation functions (1.3) and (1.6).

The total hydrodynamic forces are calculated by the formula

$$R_x - iR_y = i\int_L (p - p_\infty)\, e^{-i\theta} d\sigma, \tag{2.2}$$

where θ is the slope angle of the tangent to the contour L, which is reckoned from the axis x counterclockwise, while integration over the contour L is performed clockwise. Let us present (2.2) in the form

$$R_x - iR_y = \sum_{r=1}^{2}\sum_{m=1}^{N}(\Delta R_{xm}^{(r)} - i\Delta R_{ym}^{(r)}) \tag{2.3}$$

($\Delta R_{xm}^{(r)}$, $\Delta R_{ym}^{(r)}$ are the hydrodynamic forces acting on the mth element of the contour L_r). From (1 6) and (2.1)-(2.3) it follows that for the contour elements beyond the leading edge ($m = 2, \ldots, N$)

$$\Delta R_{xm}^{(r)} - i\Delta R_{ym}^{(r)} = -\frac{i}{2}\rho\Delta_m^{(r)}\left[\left(\frac{\Gamma_m^{(r)}}{\Delta_m^{(r)}}\right)^2 - v_\infty^2\right] e^{-i\theta_m^{(r)}} \tag{2.4}$$

($\theta_m^{(r)}$ is the angle θ at the point $z_m^{(r)}$ or at a control point at the appropriate element).

At the elements with number $m = 1$, which are adjacent to the leading edge, the intensity of the vortex layer is given by approximating function (1.3), and the profile contour, by (1.1). Therefore, at $m = 1$

$$\begin{gathered}\Delta R_{x1}^{(r)} - i\Delta R_{y1}^{(r)} = (\Delta R_{\xi 1}^{(r)} - i\Delta R_{\eta 1}^{(r)})\, e^{-i\theta_0}, \\ \Delta R_{\xi 1}^{(r)} - i\Delta R_{\eta 1}^{(r)} = \frac{1}{2}(-1)^r \int_0^{\xi_{r1}} \left\{\left[B_r + \frac{A_r}{\sqrt{\xi}\sqrt{1 + (\eta_r')^2}}\right]^2 - v_\infty^2\right\}(\eta_r' + i)\, d\xi.\end{gathered} \tag{2.5}$$

Calculating the integrals we obtain

$$\Delta R_{\xi 1}^{(r)} = \frac{1}{2}\rho\left[-|\eta_{r1}|(B_r^2 - v_\infty^2) - 8\beta_r\sqrt{\xi_{r1}}A_r B_r \ln\frac{\sqrt{1+\beta_r^2}+1}{\sqrt{1+\beta_r^2}-1} - 2A_r^2 \arctan\frac{1}{\beta_r}\right]; \tag{2.6}$$

$$\Delta R_{\eta 1}^{(r)} = -\frac{1}{2}\rho(-1)^r\left\{4\sqrt{\xi_{r1}}A_r B_r\left[\sqrt{1+\beta_r^2} - \beta_r\right] + A_r^2 \ln\frac{1+\beta_r^2}{\beta_r^2} + (B_r^2 - v_\infty^2)\,\xi_{r1}\right\} \tag{2.7}$$

$$[\beta_r = |\eta_{r1}|/(2\xi_{r1})].$$

The moment of hydrodynamic forces with respect to the axis passing through a point is calculated from the distributed forces $\Delta R^{(r)}_{xm}$ and $\Delta R^{(r)}_{ym}$. In particular, the moment with respect to the coordinate origin is

$$M = \sum_{r=1}^{2} \sum_{m=1}^{N} (\Delta R^{(r)}_{ym} x^{(r)}_m - \Delta R^{(r)}_{xm} y^{(r)}_m).$$

Here $x^{(r)}_m$ and $y^{(r)}_m$ are the coordinates of the discrete vortex $\Gamma^{(r)}_m$ (under the assumption that the forces $\Delta R^{(r)}_{xm}$ and $\Delta R^{(r)}_{ym}$ are applied to $\Gamma^{(r)}_m$).

Formulas (2.3)–(2.7) are applicable for calculating the hydrodynamic forces acting on profiles of any thickness. Let us show that in the limiting case of an infinitely thin profile the results obtained are in complete correspondence with the thin-wing theory [1].

Assume $|\eta_{r1}| \to 0$. Then the contour L is transformed into an airfoil section L_0, and the angle $\theta^{(2)}_m = \theta^{(1)}_m + \pi$; in this case L_0 is simulated by two vortex layers γ_1 and γ_2. In the thin-wing theory, as mentioned before, L_0 is simulated by one vortex layer $\gamma = \gamma_1 + \gamma_2$. Let Δ_m be the lengths of the elements of this layer, while $\Delta^{(1)}_m = \Delta^{(2)}_m = \Delta_m$ $(m = 1, \dots, N)$. It follows from (2.4) that each element of L_0 experiences a force

$$\Delta R_m = \frac{1}{2}\rho\Delta_m[(\Gamma^{(1)}_m/\Delta_m)^2 - (\Gamma^{(2)}_m/\Delta_m)^2], \quad m = 2, \dots, N, \tag{2.8}$$

directed to the normal of the element. Let us denote

$$v_0 = (v^+_\sigma + v^-_\sigma)/2, \quad \gamma = v^-_\sigma - v^+_\sigma,$$

where v^+_σ and v^-_σ are the limiting values of the tangential fluid velocity component as L_0 is approached from above and below. Within the framework of approximation (1.6) at the mth element

$$v^+_\sigma = -\Gamma^{(1)}_m/\Delta_m, \quad v^-_\sigma = \Gamma^{(2)}_m/\Delta_m, \quad v_0 = (\Gamma^{(2)}_m - \Gamma^{(1)}_m)/(2\Delta_m), \quad \gamma = (\Gamma^{(1)}_m + \Gamma^{(2)}_m)/\Delta_m. \tag{2.9}$$

Taking into account (2.9), formula (2.8) takes the form of the Zhukovskii theorem "in the small"

$$\Delta R_m/\Delta_m = -\rho v_0 \gamma, \quad m = 2, \dots, N.$$

Let us consider the hydrodynamic forces acting on the first element of the airfoil section L_0. Following the thin-wing theory, this element is affected by two forces. One force is directed normally and is determined by the Zhukovskii theorem "in the small," and the other, which is called the suction force, is applied to the leading edge and is tangent to L_0. With variation of the intensity of the vortex layer in the vicinity of the leading edge L_0 according to the law (1.5) the suction force is

$$Q = -\pi\rho A^2/4. \tag{2.10}$$

Let us determine the form taken by formulas (2.6) and (2.7) in the limiting case of an infinitely thin profile. We note that the forces $\Delta R^{(r)}_{\xi 1}$ $(r = 1, 2)$ are tangent to L_0 at the leading edge, while $\Delta R^{(r)}_{\eta 1}$, is directed to the normal. At $|\eta_{r1}| \to 0$ the coefficient $\beta_r \to 0$ and (2.6) takes the form

$$\Delta R^{(r)}_{\xi 1} = -\pi\rho A_r^2/2, \quad r = 1, 2.$$

The total force acting on the first element of the airfoil section L_0 in the direction of the tangent is

$$\Delta R_{\xi 1} = \Delta R^{(1)}_{\xi 1} + \Delta R^{(2)}_{\xi 1} = -\frac{\pi}{2}\rho(A_1^2 + A_2^2).$$

Expressing A_1, A_2 with the help of (1.3) in terms of the intensities of the discrete vortices $\Gamma^{(1)}_1$ and $\Gamma^{(2)}_1$, we obtain

$$\Delta R_{\xi 1} = -\frac{\pi}{16}\rho\frac{(\Gamma^{(1)}_1 + \Gamma^{(2)}_1)^2}{\Delta_1}. \tag{2.11}$$

Formula (2.11) is in complete agreement with (2.10). Actually, by definition

$$\Gamma_1^{(1)} + \Gamma_1^{(2)} = \int_0^{\Delta_1} \gamma(\xi)\, d\xi = 2\sqrt{\Delta_1}A.$$

Hence

$$a = (\Gamma_1^{(1)} + \Gamma_1^{(2)})/(2\sqrt{\Delta_1}),$$

which results in the coincidence of (2.10) and (2.11). Similarly, using (2.7) to estimate the normal force $\Delta R_{\eta 1} = \Delta R_{\eta 1}^{(1)} + \Delta R_{\eta 1}^{(2)}$ acting on the first element L_0, we arrive at (2.8).

Thus, in the limiting case of an infinitely thin profile, formulas (2.4)–(2.7) derived on the basis of approximating functions (1.1)–(1.3) are in complete agreement with the results of the thin-wing theory.

The algorithm developed was tested in the problem of motion of an airfoil near a screen [2].

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